# CEE598 - Visual Sensing for Civil Infrastructure Eng. & Mgmt.

## Session 4 – Camera Calibration

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# Outline

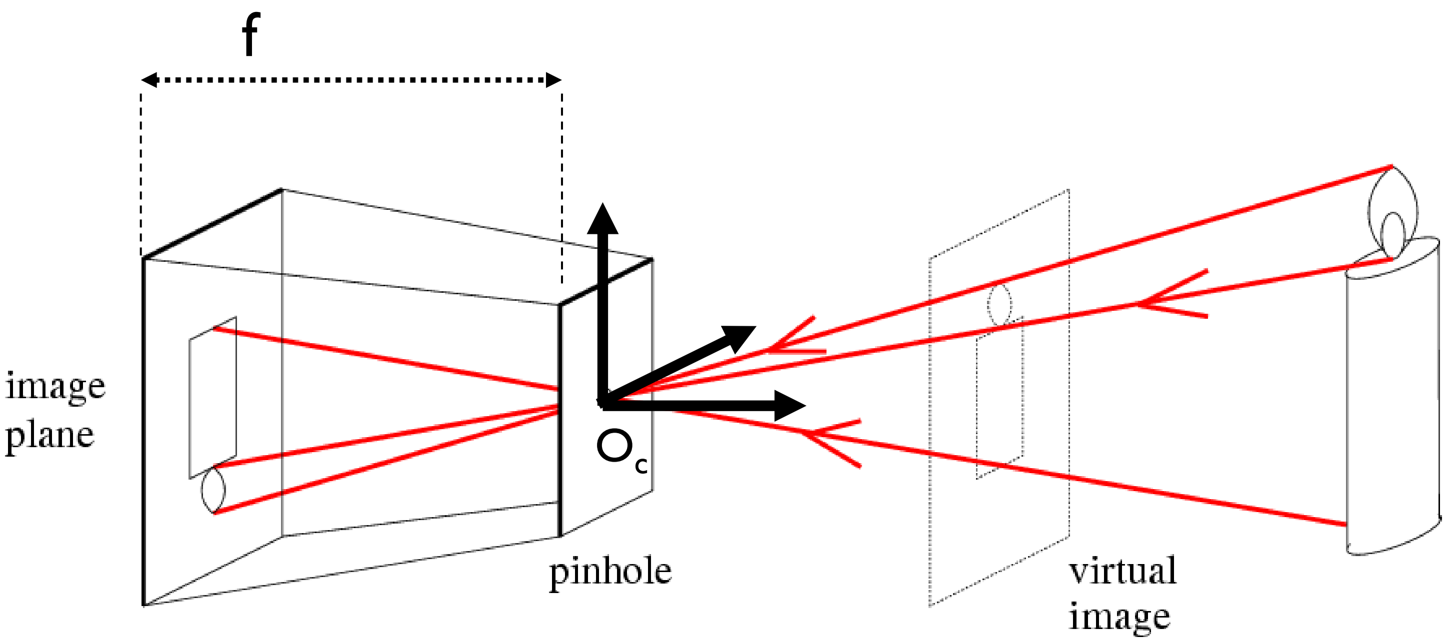
## Camera Calibration

- Review Camera Parameters
- Camera Calibration Problem
- Example

Reading: [FP] Chapter 3
[HZ] Chapter 7
[S] Chapter 6

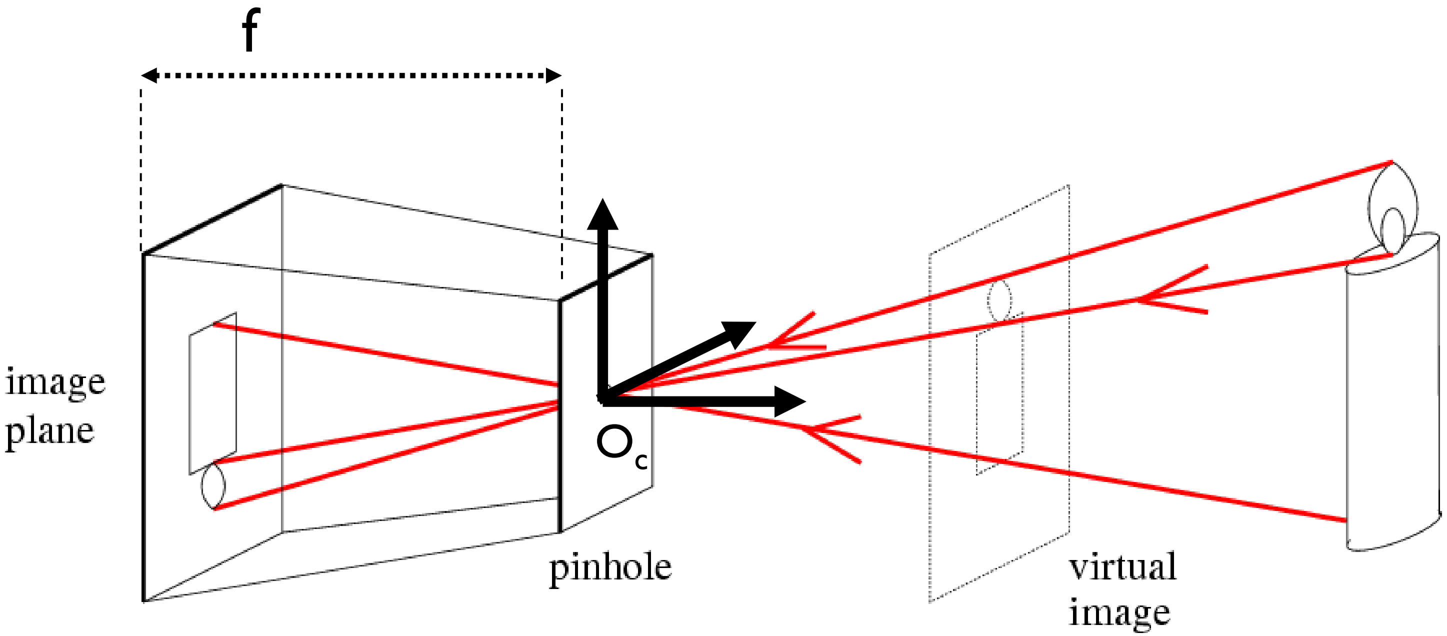
Some slides in this lecture are courtesy to Profs. S. Savarese, J. Ponce & F-F Li

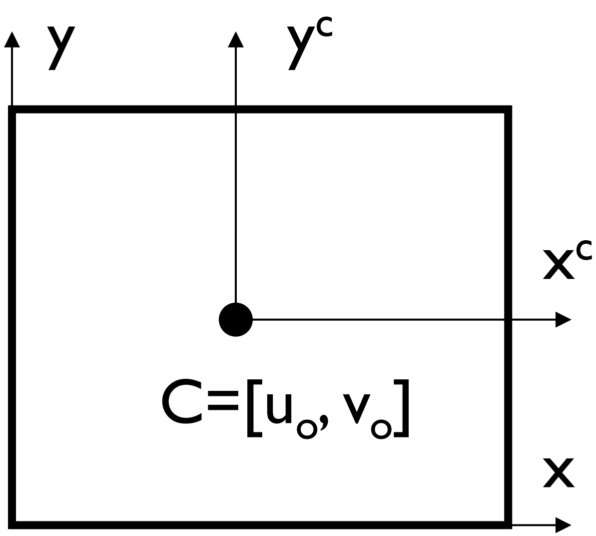
# Projective camera




f = focal length

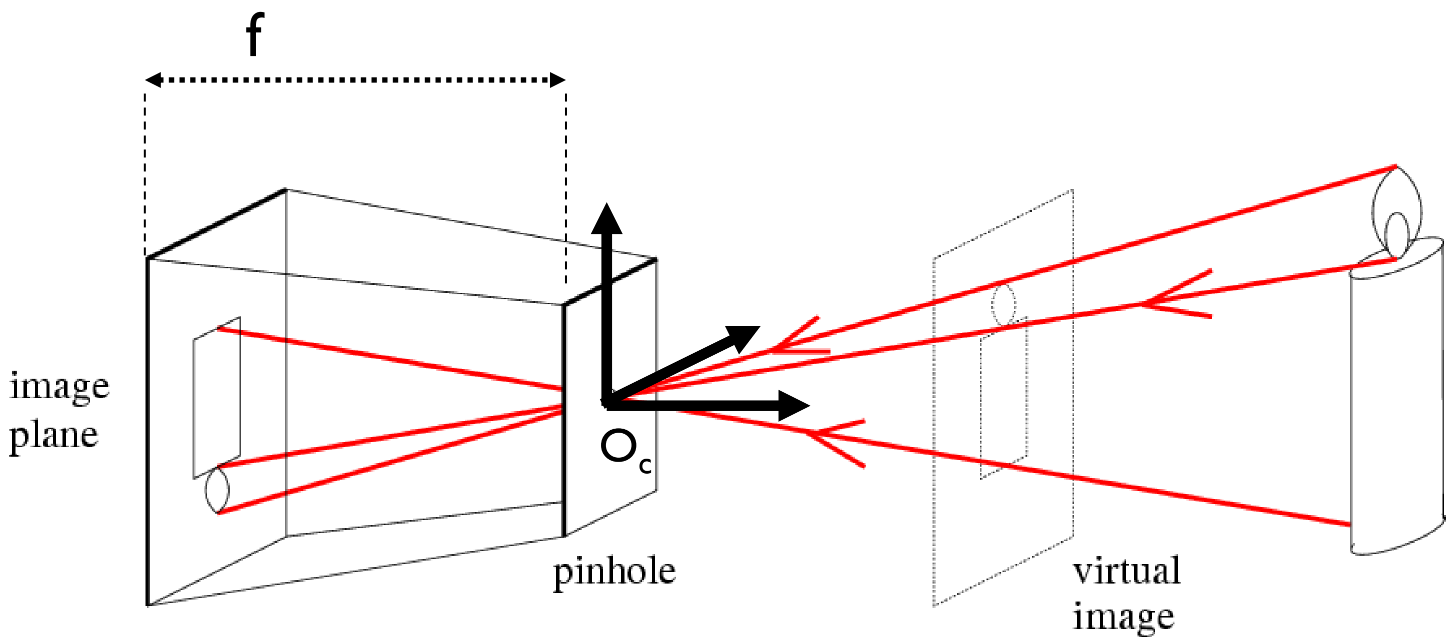
# Projective camera




f = focal length
$u_o, v_o$ = offset

# Projective camera




Units: k,l [pixel/m]

f [m]

Non-square pixels

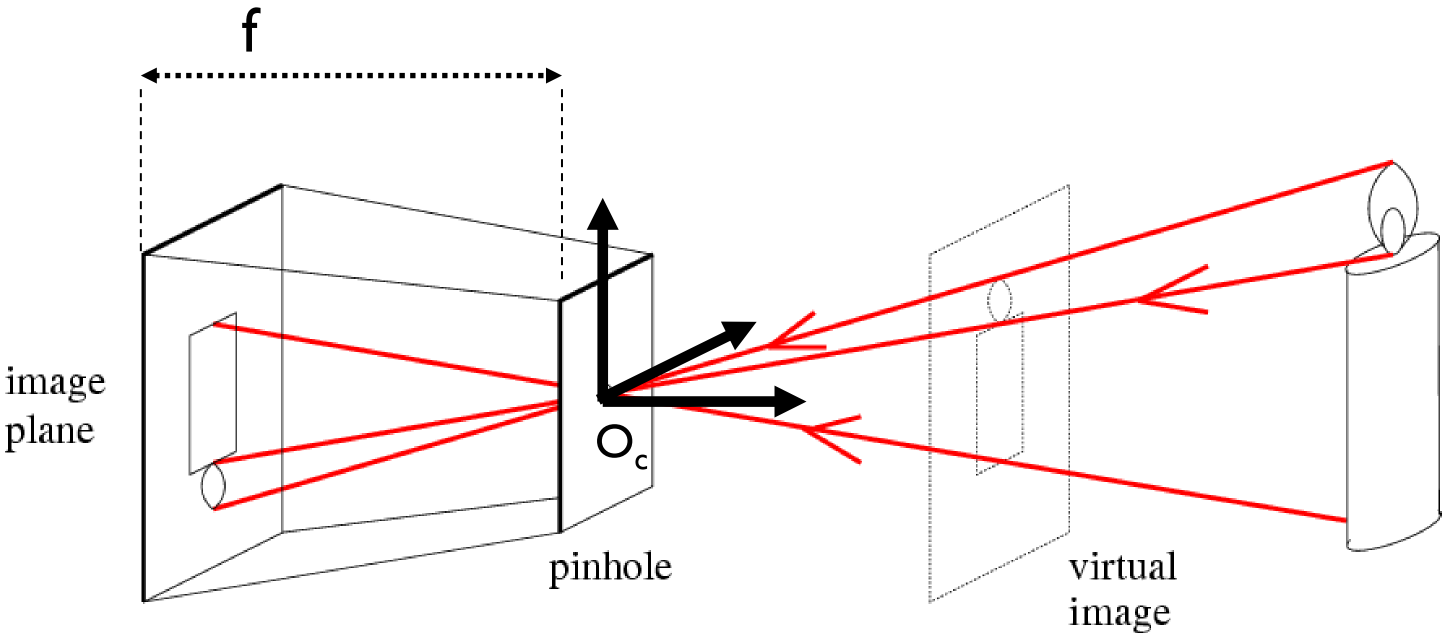
$\alpha, \beta$ [pixel]

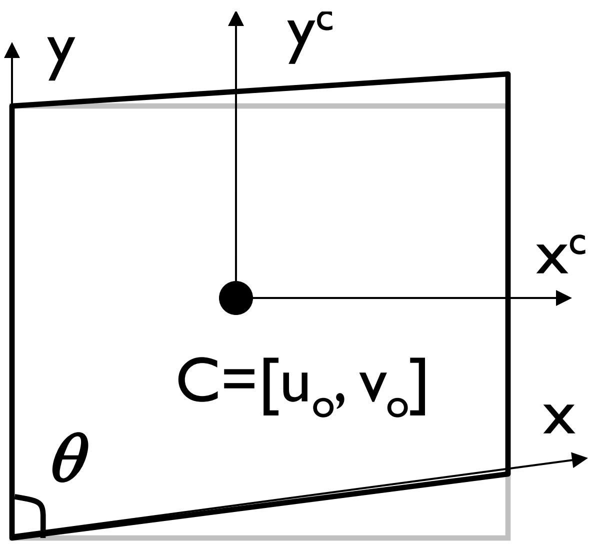
f = focal length

$u_o, v_o$ = offset

$\alpha, \beta \rightarrow$ non-square pixels

# Projective camera



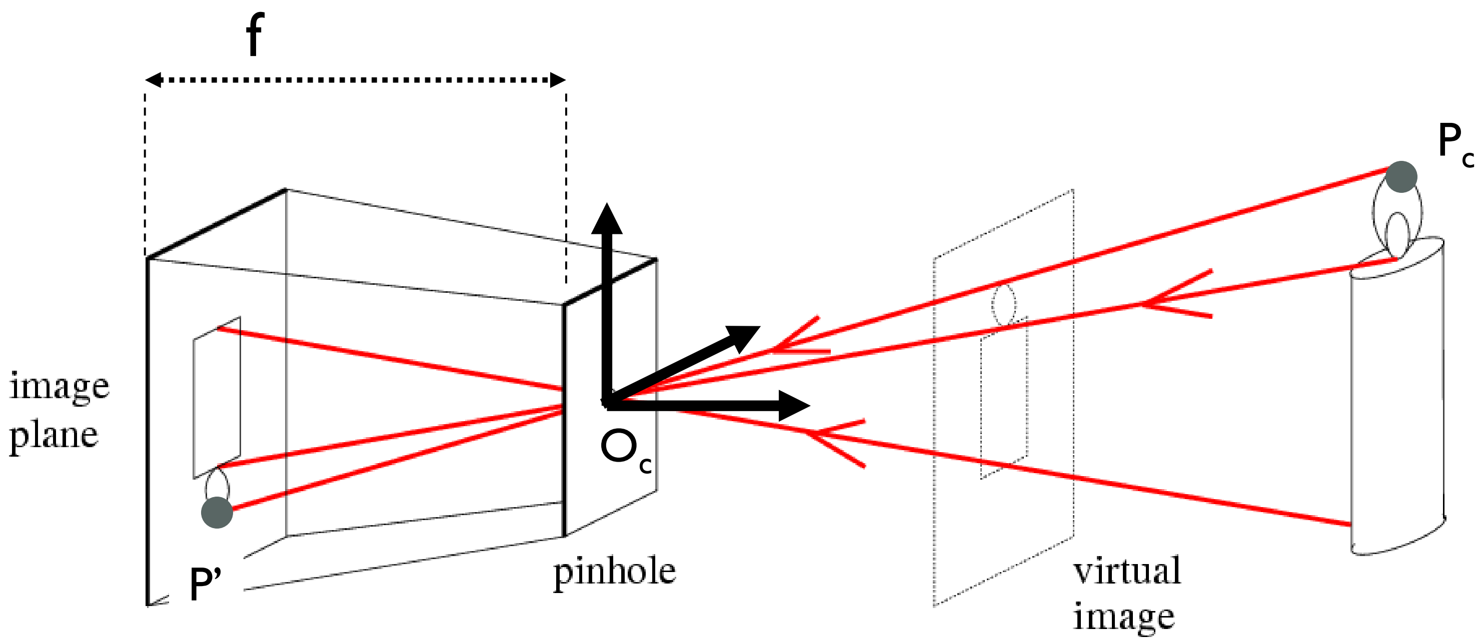

f = focal length

$u_o, v_o$ = offset

$\alpha, \beta \rightarrow$ non-square pixels

$\theta$ = skewness

# Projective camera




$$P' = \begin{bmatrix} \alpha & -\alpha\cot\theta & u_o & 0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_o & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

K has 5 degrees of freedom!

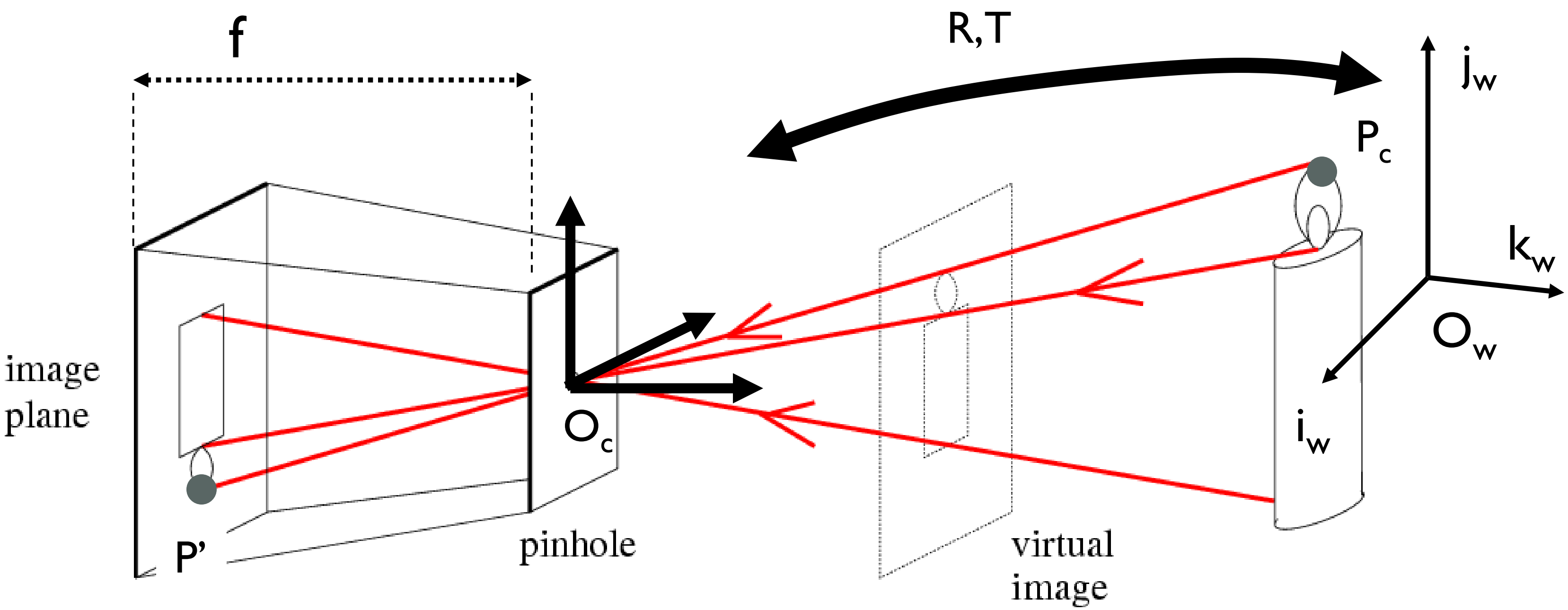
f = focal length

$u_o$, $v_o$ = offset

$\alpha, \beta \rightarrow$ non-square pixels

$\theta$ = skewness

# Projective camera




$$P_c = [R \quad T] P_w$$

f = focal length

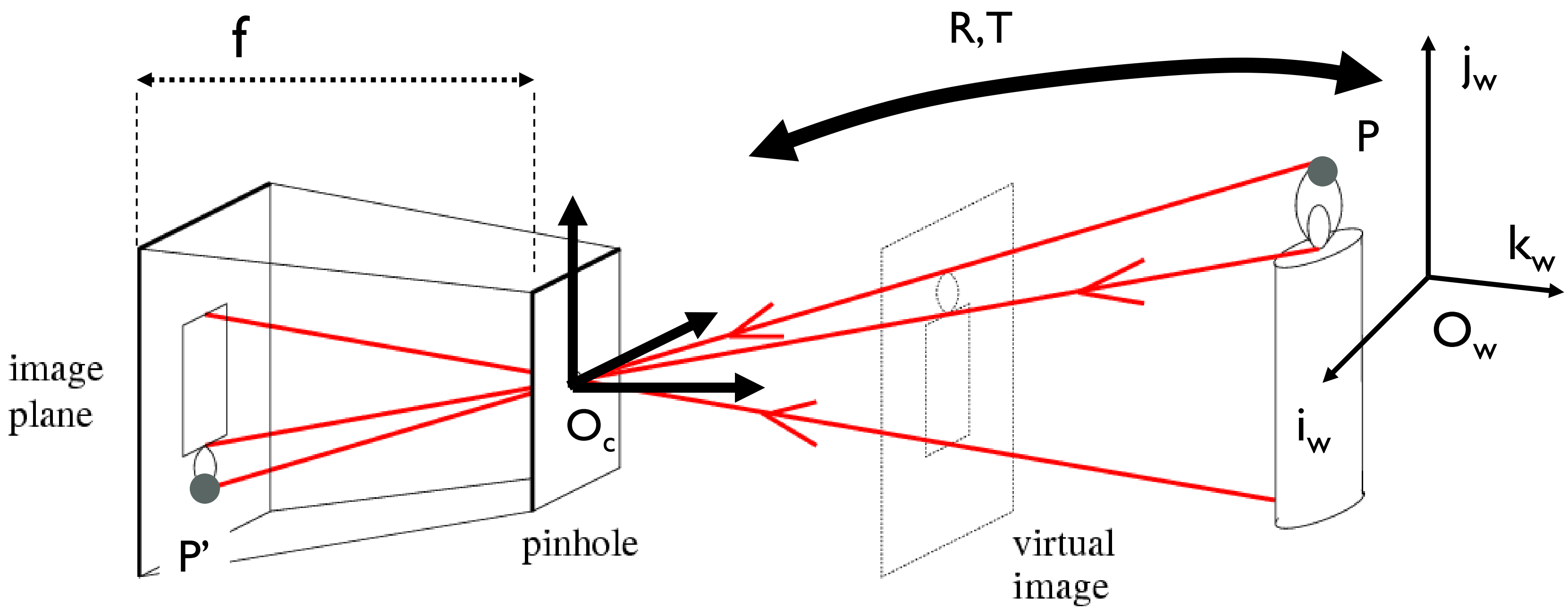
$u_o, v_o$ = offset

$\alpha, \beta \rightarrow$ non-square pixels

$\theta$ = skewness

R,T = rotation, translation

# Projective camera




$$P' = M\,P_w$$

$$= K\begin{bmatrix} R & T \end{bmatrix} P_w$$

Internal parameters

External parameters

f = focal length

$u_o, v_o$ = offset

$\alpha, \beta \rightarrow$ non-square pixels

$\theta$ = skewness

R,T = rotation, translation

# Goal of calibration

- Estimate intrinsic and extrinsic parameters
- from 1 or multiple images

$$P' = M\,P_w \quad = K\begin{bmatrix} R & T \end{bmatrix} P_w$$

Internal parameters

External parameters

# Goal of calibration

- Estimate intrinsic and extrinsic parameters
- from 1 or multiple images

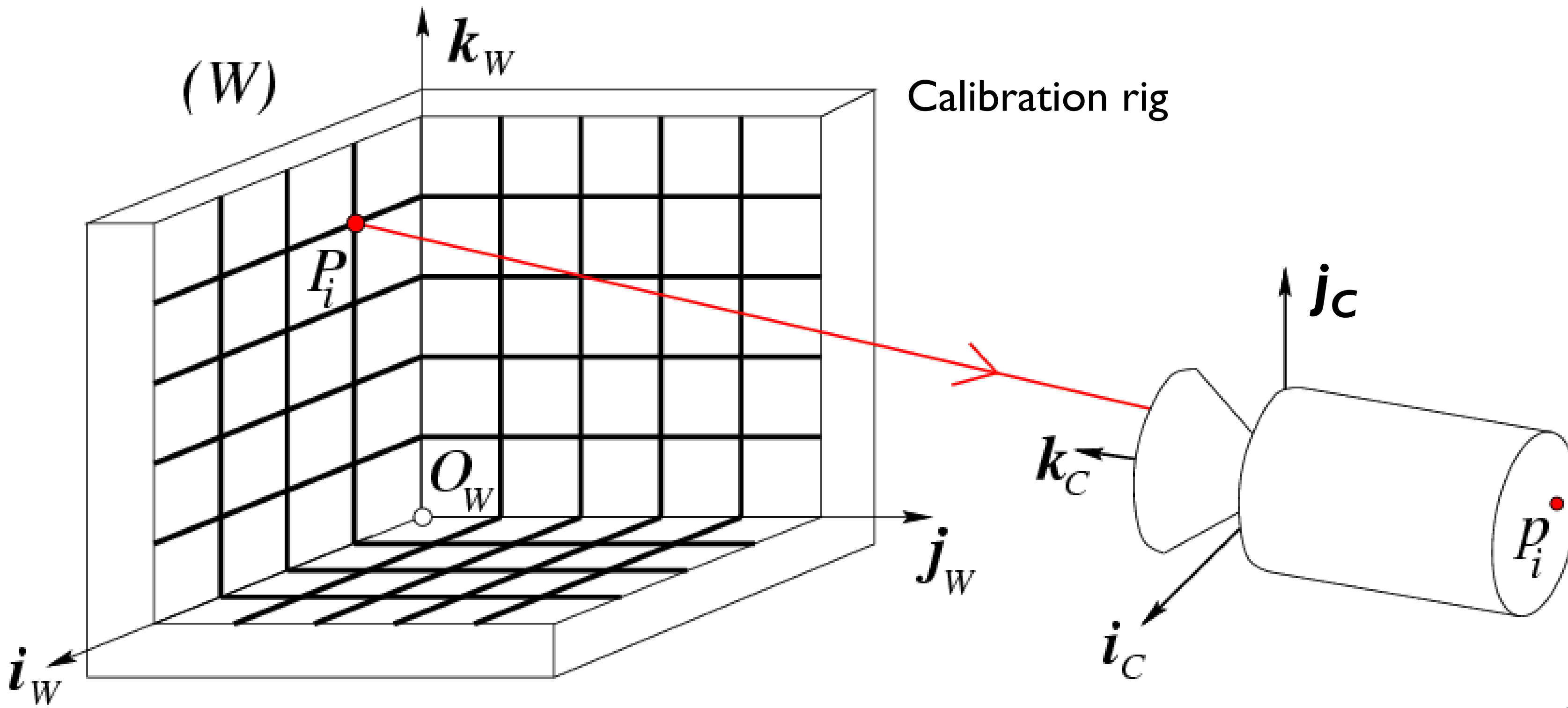
$$P' = M\,P_w = K\begin{bmatrix} R & T \end{bmatrix} P_w$$

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

$$K = \begin{bmatrix} \alpha & -\alpha \cot\theta & u_o \\ 0 & \dfrac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

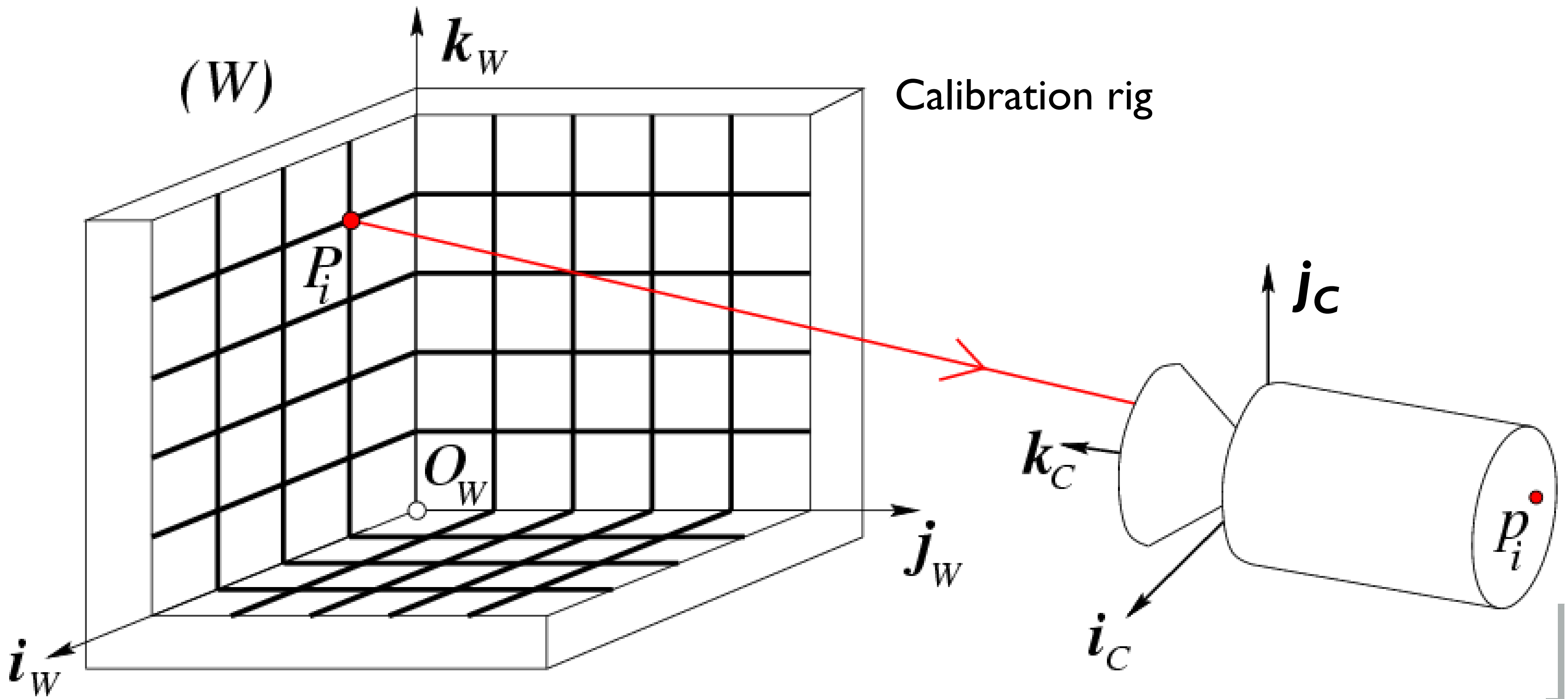
Note: To simplify notation let $P = P_w$

# Calibration Problem




•$P_1 \ldots P_n$ with known positions in $[O_w, i_w, j_w, k_w]$
•$p_1, \ldots p_n$ known positions in the image
Goal: compute intrinsic and extrinsic parameters

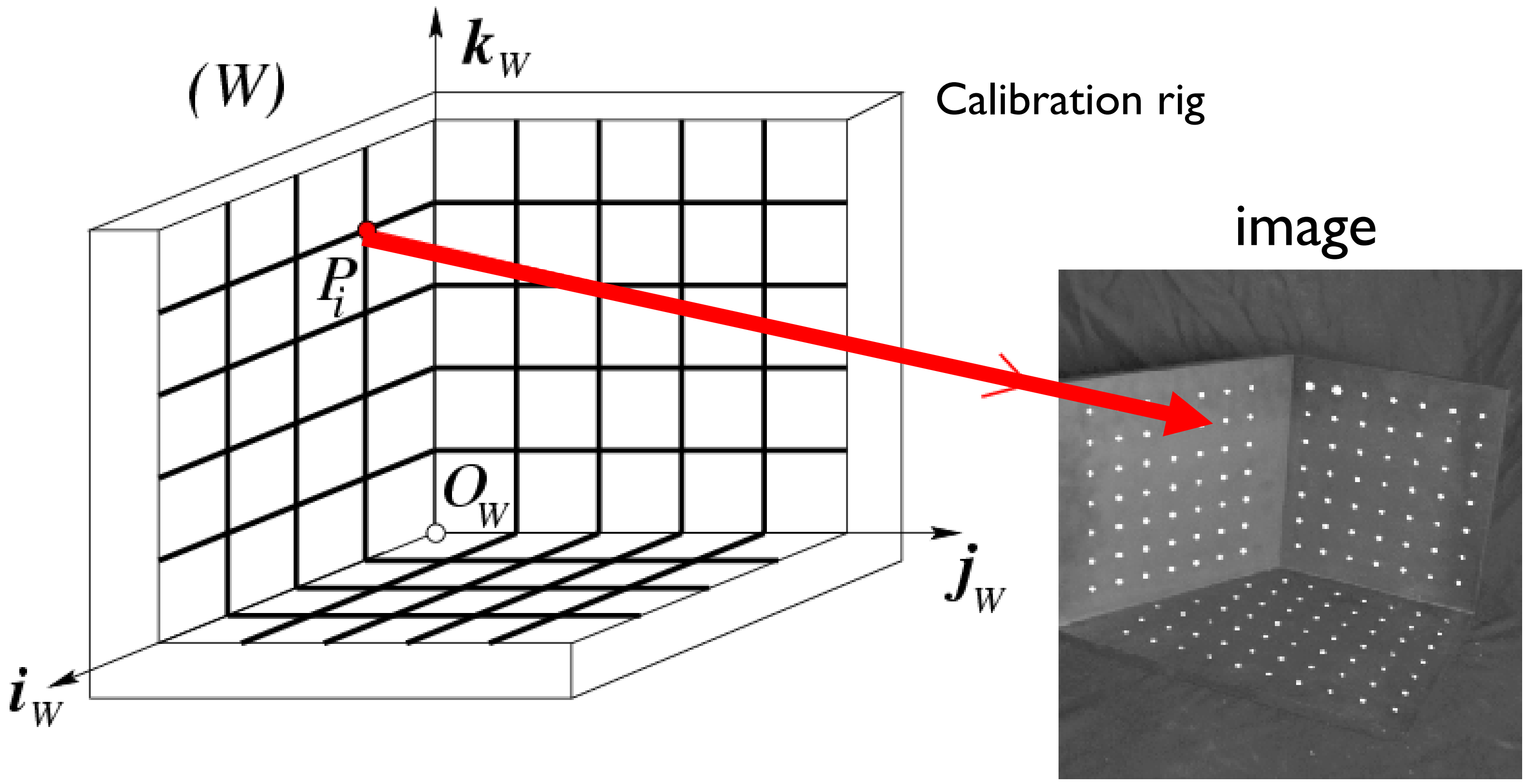
# Calibration Problem




## How many correspondences do we need?

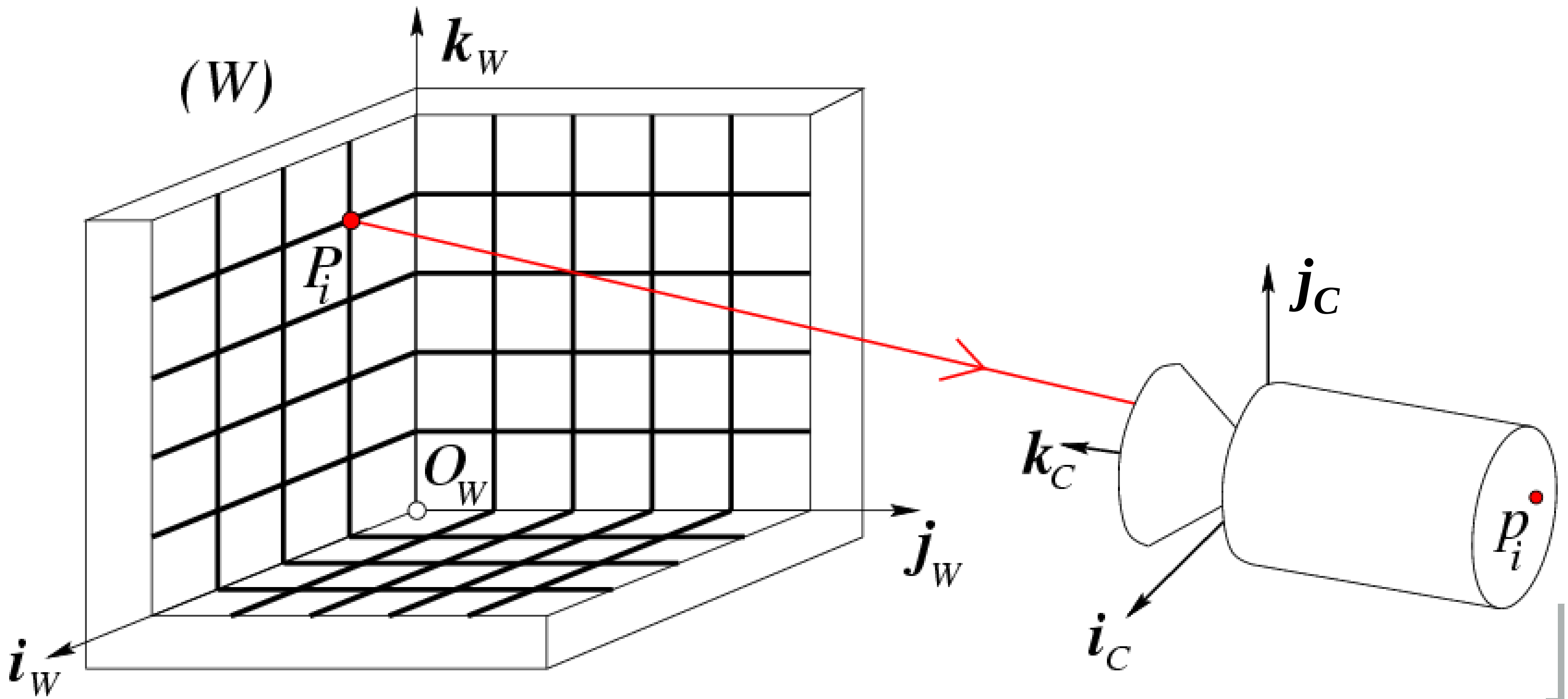
- P has 11 unknown
- We need 11 equations
- 6 correspondences would do it

# Calibration Problem




In practice: user may need to look at the image and select the n>=6 correspondences

# Calibration Problem




$$P_i \rightarrow M\,P_i \rightarrow p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{m}_1\,P_i}{\mathbf{m}_3\,P_i} \\ \dfrac{\mathbf{m}_2\,P_i}{\mathbf{m}_3\,P_i} \end{bmatrix} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

in pixels

# Calibration Problem

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{m}_1 \, P_i}{\mathbf{m}_3 \, P_i} \\ \dfrac{\mathbf{m}_2 \, P_i}{\mathbf{m}_3 \, P_i} \end{bmatrix}$$

$$u_i = \frac{\mathbf{m}_1 \, P_i}{\mathbf{m}_3 \, P_i} \rightarrow u_i(\mathbf{m}_3 \, P_i) = \mathbf{m}_1 \, P_i$$

$$\rightarrow \begin{pmatrix} \mathbf{m}_1 - u_i \mathbf{m}_3 \\ \mathbf{m}_2 - v_i \mathbf{m}_3 \end{pmatrix} P_i = 0$$

$$v_i = \frac{\mathbf{m}_2 \, P_i}{\mathbf{m}_3 \, P_i} \rightarrow v_i(\mathbf{m}_3 \, P_i) = \mathbf{m}_2 \, P_i$$

# Calibration Problem

$$\begin{cases} \begin{pmatrix} \mathbf{m}_1 - u_1\mathbf{m}_3 \\ \mathbf{m}_2 - v_1\mathbf{m}_3 \end{pmatrix} P_1 = 0 \\ \vdots \\ \begin{pmatrix} \mathbf{m}_1 - u_i\mathbf{m}_3 \\ \mathbf{m}_2 - v_i\mathbf{m}_3 \end{pmatrix} P_i = 0 \\ \vdots \\ \begin{pmatrix} \mathbf{m}_1 - u_n\mathbf{m}_3 \\ \mathbf{m}_2 - v_n\mathbf{m}_3 \end{pmatrix} P_n = 0 \end{cases}$$

# Review on Block Matrix Multiplication

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{bmatrix}$$

What is *AB* ?

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

# Calibration Problem

$$
\begin{cases}
\quad \vdots \\
\begin{pmatrix} \mathbf{m}_1 - u_i \mathbf{m}_3 \\ \mathbf{m}_2 - v_i \mathbf{m}_3 \end{pmatrix} P_i = 0 \\
\quad \vdots \\
\begin{pmatrix} \mathbf{m}_1 - u_n \mathbf{m}_3 \\ \mathbf{m}_2 - v_n \mathbf{m}_3 \end{pmatrix} P_n = 0
\end{cases}
\longrightarrow \quad \mathcal{P}\boldsymbol{m} = 0
$$

known ($\mathcal{P}$), unknown ($\boldsymbol{m}$)

Homogenous linear system

$$
\mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix}
\boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1 \boldsymbol{P}_1^T \\
\boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1 \boldsymbol{P}_1^T \\
\cdots & \cdots & \cdots \\
\boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n \boldsymbol{P}_n^T \\
\boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n \boldsymbol{P}_n^T
\end{pmatrix}_{2n \times 12}
$$

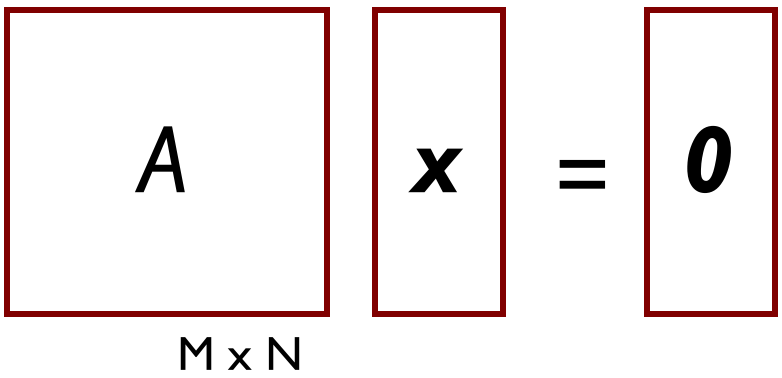
1x4 ($\boldsymbol{P}_1^T$)

$$
\boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{pmatrix}_{12 \times 1}
$$

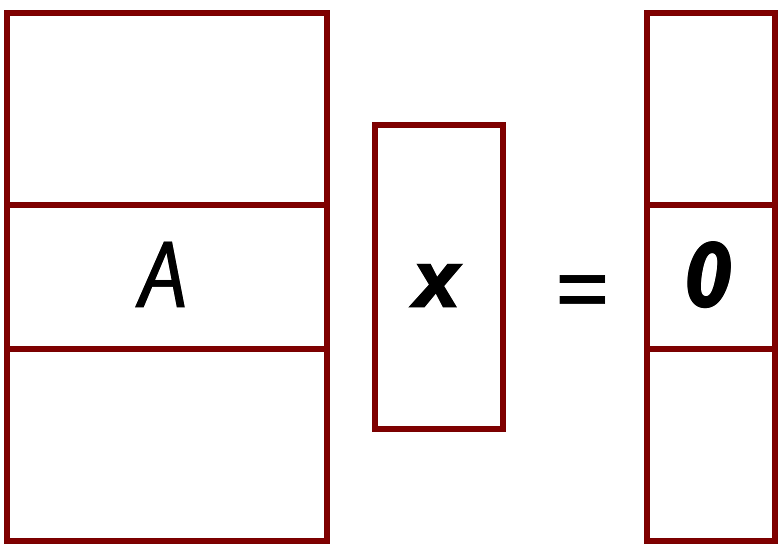
4x1 ($\mathbf{m}_1^T$)

# Homogeneous M x N Linear Systems

M=number of equations
N=number of unknown




Square system (M=N):




Rectangular system (M>N)

- 0 is always a solution
- Rank(A) = N
- noisy measurements

Minimize $|Ax|^2$

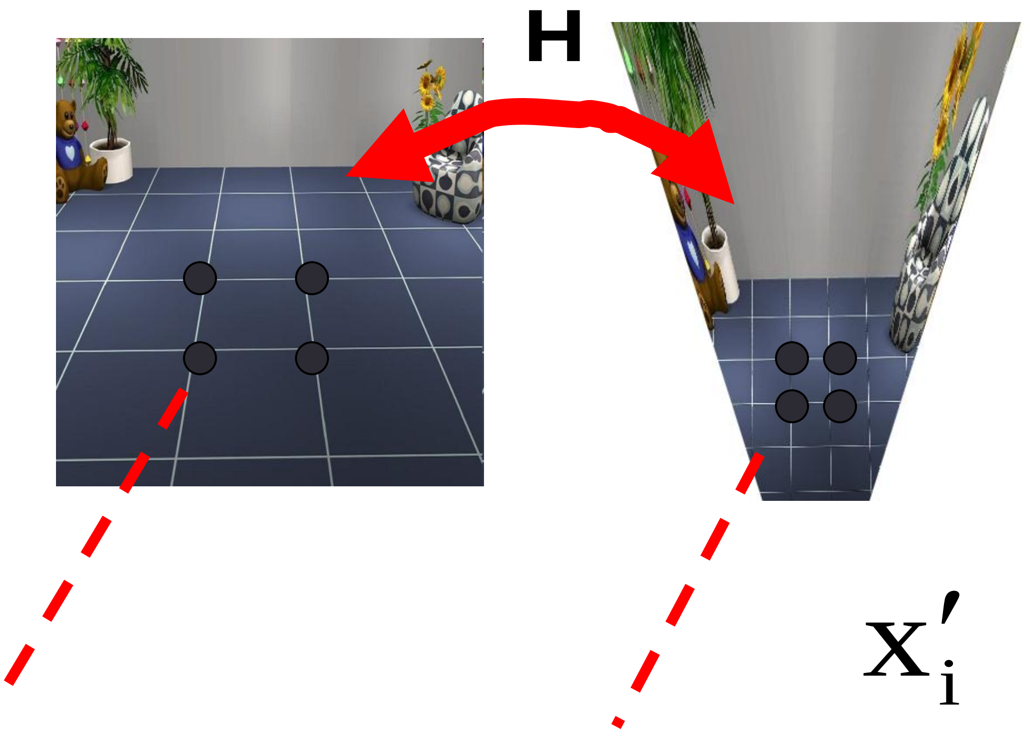
under the constraint $|x|^2 = 1$

# Calibration Problem

$$\mathcal{P}\boldsymbol{m} = 0 \; .$$

How do we solve this homogenous linear system?

# DLT algorithm (Direct Linear Transformation)




$x_i$ $\quad$ $x'_i$

$$x'_i = H x_i$$

$$x'_i \times H x_i = 0 \longrightarrow \underbrace{A_i}_{\text{Function of measurements}} \overbrace{\mathbf{h}}^{\text{unknown}} = 0$$

# General Calibration Problem

$$\mathcal{P}\boldsymbol{m} = 0 \, .$$

↓

$$U_{2n\times12}\; D_{\,12\times12} V^{T}{}_{12\times12}$$

Last column of V gives $\boldsymbol{m}$

↓

$$M\, P_i \rightarrow p_i$$

# Extracting camera parameters

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \frac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \frac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix} = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}$$

$A$ $\qquad$ $\mathbf{b}$

$$\mathrm{K} = \begin{bmatrix} \alpha & -\alpha\cot\theta & u_o \\ 0 & \frac{\beta}{\sin\theta} & v_o \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$$

Estimated values

## Intrinsic

$$\rho = \frac{\pm 1}{|\mathbf{a}_3|} \qquad \mathrm{u}_o = \rho^2 (\mathbf{a}_1 \cdot \mathbf{a}_2)$$

$$\mathrm{v}_o = \rho^2 (\mathbf{a}_2 \cdot \mathbf{a}_3)$$

$$\cos\theta = \frac{(\mathbf{a}_1 \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}{|\mathbf{a}_1 \times \mathbf{a}_3| \cdot |\mathbf{a}_2 \times \mathbf{a}_3|}$$

# Theorem (Faugeras, 1993)

Let $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ be a $3 \times 4$ matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

# Extracting camera parameters

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix} = \mathrm{K}\begin{bmatrix} \mathrm{R} & \mathrm{T} \end{bmatrix}$$

$A$ $\quad$ $\mathbf{b}$

$$\mathrm{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$$

Estimated values

## Intrinsic

$$\alpha = \rho^2 |\mathbf{a}_1 \times \mathbf{a}_3| \sin\theta$$

$$\beta = \rho^2 |\mathbf{a}_2 \times \mathbf{a}_3| \sin\theta$$

$\rightarrow$ f

# Extracting camera parameters

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \frac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \frac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix} = \mathrm{K}[\mathrm{R} \quad \mathrm{T}]$$

$\rho\,\mathrm{A}$ $\qquad$ $\rho\,\mathbf{b}$

$$\mathrm{A} = \begin{bmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_3^{\mathrm{T}} \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \mathrm{b}_3 \end{bmatrix}$$

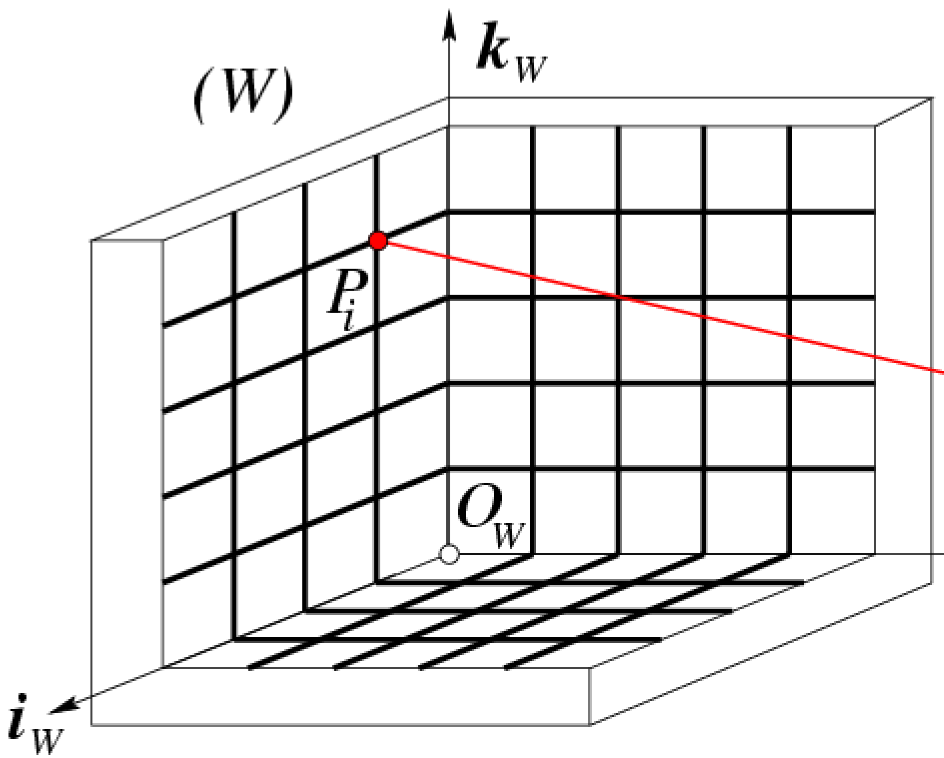
Estimated values

## Extrinsic

$$\mathbf{r}_1 = \frac{(\mathbf{a}_2 \times \mathbf{a}_3)}{|\mathbf{a}_2 \times \mathbf{a}_3|} \qquad \mathbf{r}_3 = \frac{\pm 1}{|\mathbf{a}_3|}$$

$$\mathbf{r}_2 = \mathbf{r}_3 \times \mathbf{r}_1 \qquad \mathrm{T} = \rho\,\mathrm{K}^{-1}\mathbf{b}$$

# Degenerate cases



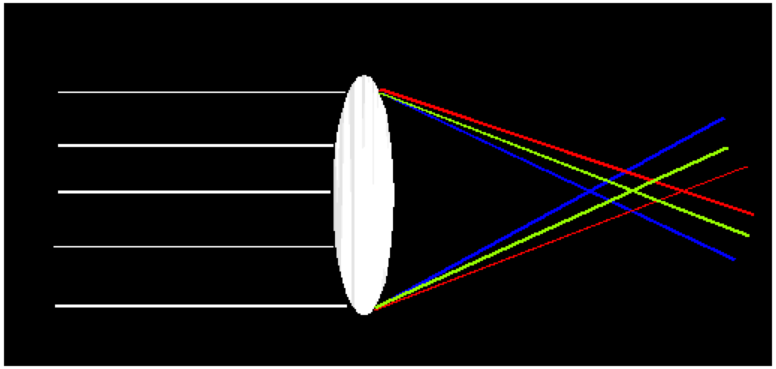

i

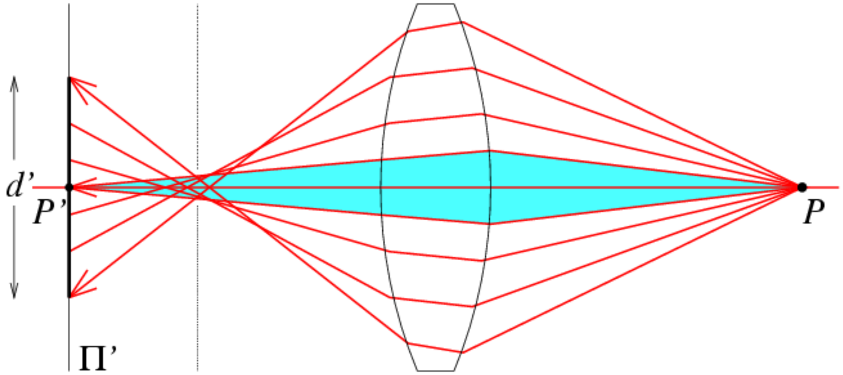
$$\mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1\boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \cdots & \cdots & \cdots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n\boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix}$$

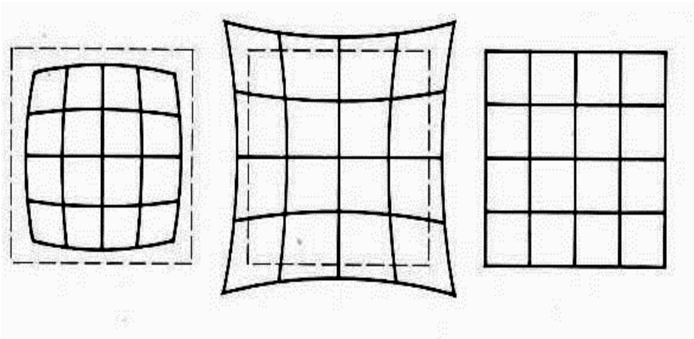
- $P_i$'s cannot lie on the same plane!
- Points cannot lie on the intersection curve of two quadric surfaces

# Taking lens distortions into account

- Chromatic Aberration
- Spherical Aberration
- Radial Distortion

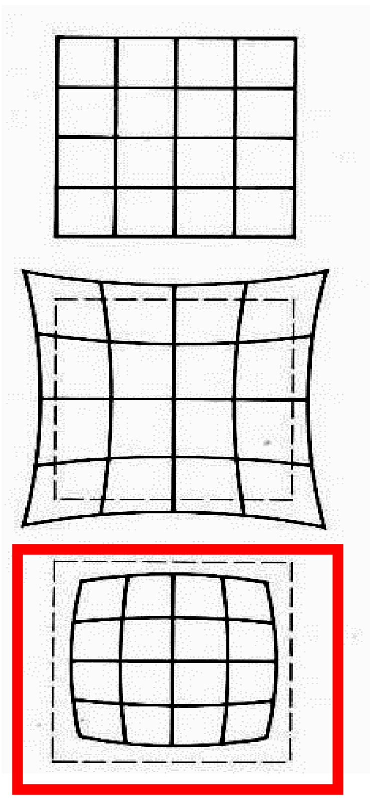




# Radial Distortion

- Caused by imperfect lenses
- Deviations are most noticeable for rays that pass through the edge of the lens

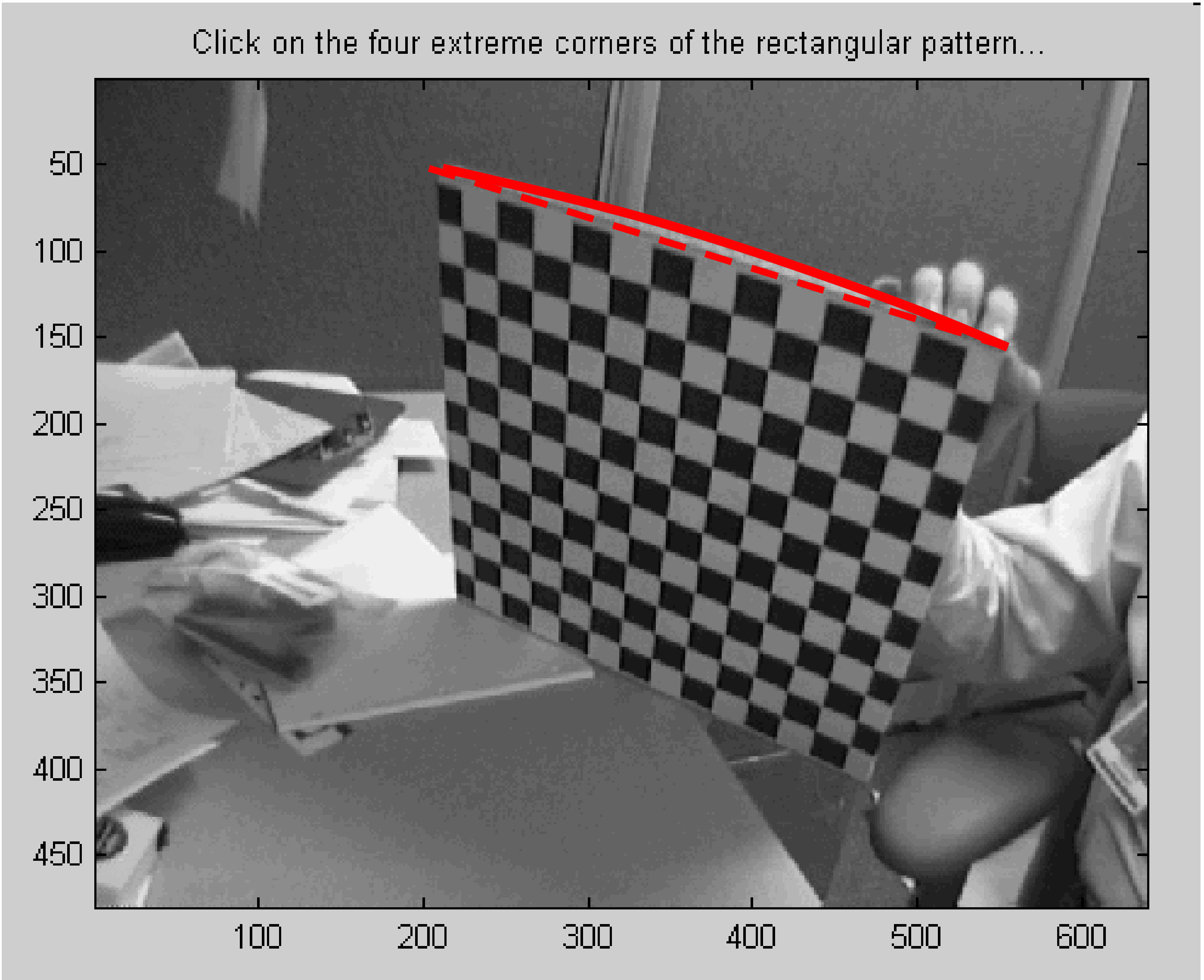


No distortion

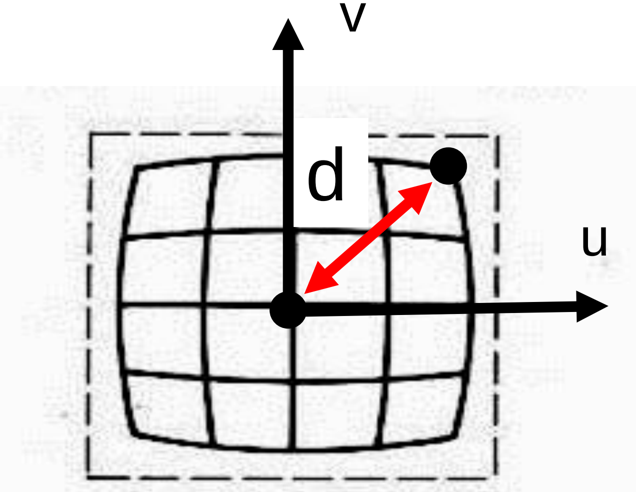
Pin cushion

Barrel

# Radial Distortion



# Radial Distortion




$$\begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix} M\, P_i \rightarrow \begin{bmatrix} u_i \\ v_i \end{bmatrix} = p_i$$

Distortion coefficient

$$d^2 = a\,u^2 + b\,v^2 + c\,u\,v \qquad \lambda = 1 \pm \sum_{p=1}^{3} \kappa_p d^{2p}$$

To model radial behavior

Polynomial function

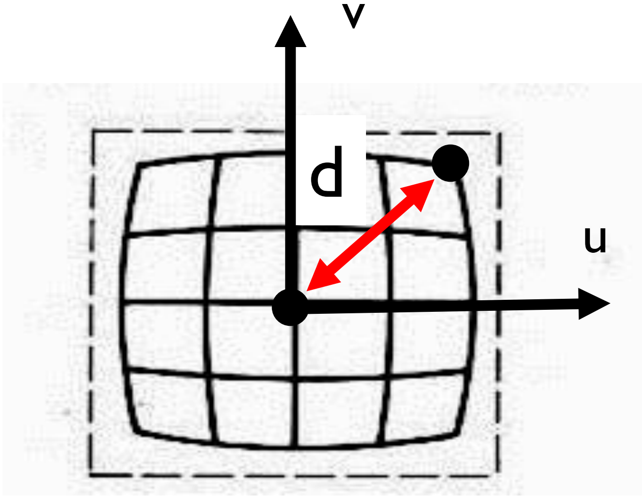
# Radial Distortion

Estimating $m_1$ and $m_2$…

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \dfrac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \dfrac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

How to do that?

Hint:




$$\frac{u_i}{v_i} = \text{slope}$$

# Radial Distortion

Estimating $m_1$ and $m_2$…

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \dfrac{\mathbf{m}_1 \, P_i}{\mathbf{m}_3 \, P_i} \\ \dfrac{\mathbf{m}_2 \, P_i}{\mathbf{m}_3 \, P_i} \end{bmatrix} \qquad \frac{u_i}{v_i} = \frac{\dfrac{(\mathbf{m}_1 \, P_i)}{(\mathbf{m}_3 \, P_i)}}{\dfrac{(\mathbf{m}_2 \, P_i)}{(\mathbf{m}_3 \, P_i)}} = \frac{\mathbf{m}_1 \, P_i}{\mathbf{m}_2 \, P_i}$$

$$\begin{cases} v_1(\mathbf{m}_1 \, P_1) - u_1(\mathbf{m}_2 \, P_1) = 0 \\ v_i(\mathbf{m}_1 \, P_i) - u_i(\mathbf{m}_2 \, P_i) = 0 \\ \vdots \\ v_n(\mathbf{m}_1 \, P_n) - u_n(\mathbf{m}_2 \, P_n) = 0 \end{cases} \qquad \mathrm{Q}\,\mathbf{n} = 0 \qquad \mathbf{n} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix}$$

Tsai technique [87]

# Radial Distortion

Once that $\mathbf{m}_1$ and $\mathbf{m}_2$ are estimated…

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} \dfrac{\mathbf{m}_1 P_i}{\mathbf{m}_3 P_i} \\ \dfrac{\mathbf{m}_2 P_i}{\mathbf{m}_3 P_i} \end{bmatrix}$$

$\mathbf{m}_3$ is non linear function of $\mathbf{m}_1$, $\mathbf{m}_2$, $\lambda$

There are some degenerate configurations for which $m_1$ and $m_2$ cannot be computed

# Radial Distortion

$$\begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix} M P_i \rightarrow \begin{bmatrix} u_i \\ v_i \end{bmatrix} = p_i \qquad Q = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{bmatrix}$$

Q

$$p_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{q}_1 P_i}{\mathbf{q}_3 P_i} \\ \frac{\mathbf{q}_2 P_i}{\mathbf{q}_3 P_i} \end{bmatrix}$$

Non-linear system of equations

$$\begin{cases} u_i \mathbf{q}_3 P_i = \mathbf{q}_1 P_i \\ v_i \mathbf{q}_3 P_i = \mathbf{q}_2 P \end{cases}$$

# Radial Distortion

- Conversion from distorted coordinates to undistorted coordinates

  - $\mathbf{p}' = f * r(\mathbf{p}) * \mathbf{p}$

  (conversion to pixel coordinates)

  $r(\mathbf{p})$ is a function that computes a scaling factor to undo the radial distortion:

  - $r(\mathbf{p}) = 1.0 + k1 * ||\mathbf{p}||^2 + k2 * ||\mathbf{p}||^4$.

# General Calibration Problem

$$X = f(P)$$

f( ) is nonlinear

measurement → $X$

parameter → $P$

-Newton Method
-Levenberg-Marquardt Algorithm

- Iterative, starts from initial solution
- May be slow if initial solution far from real solution
- Estimated solution may be function of the initial solution
- Newton requires the computation of J, H
- Levenberg-Marquardt doesn't require the computation of H

# General Calibration Problem

$$X = f(P)$$

f( ) is nonlinear

measurement

parameter

# A possible algorithm

1. Solve linear part of the system to find approximated solution
2. Use this solution as initial condition for the full system
3. Solve full system using Newton or L.M.

# General Calibration Problem

$$X = f(P)$$

f( ) is nonlinear

measurement → $X$

parameter → $P$

# Typical assumptions:

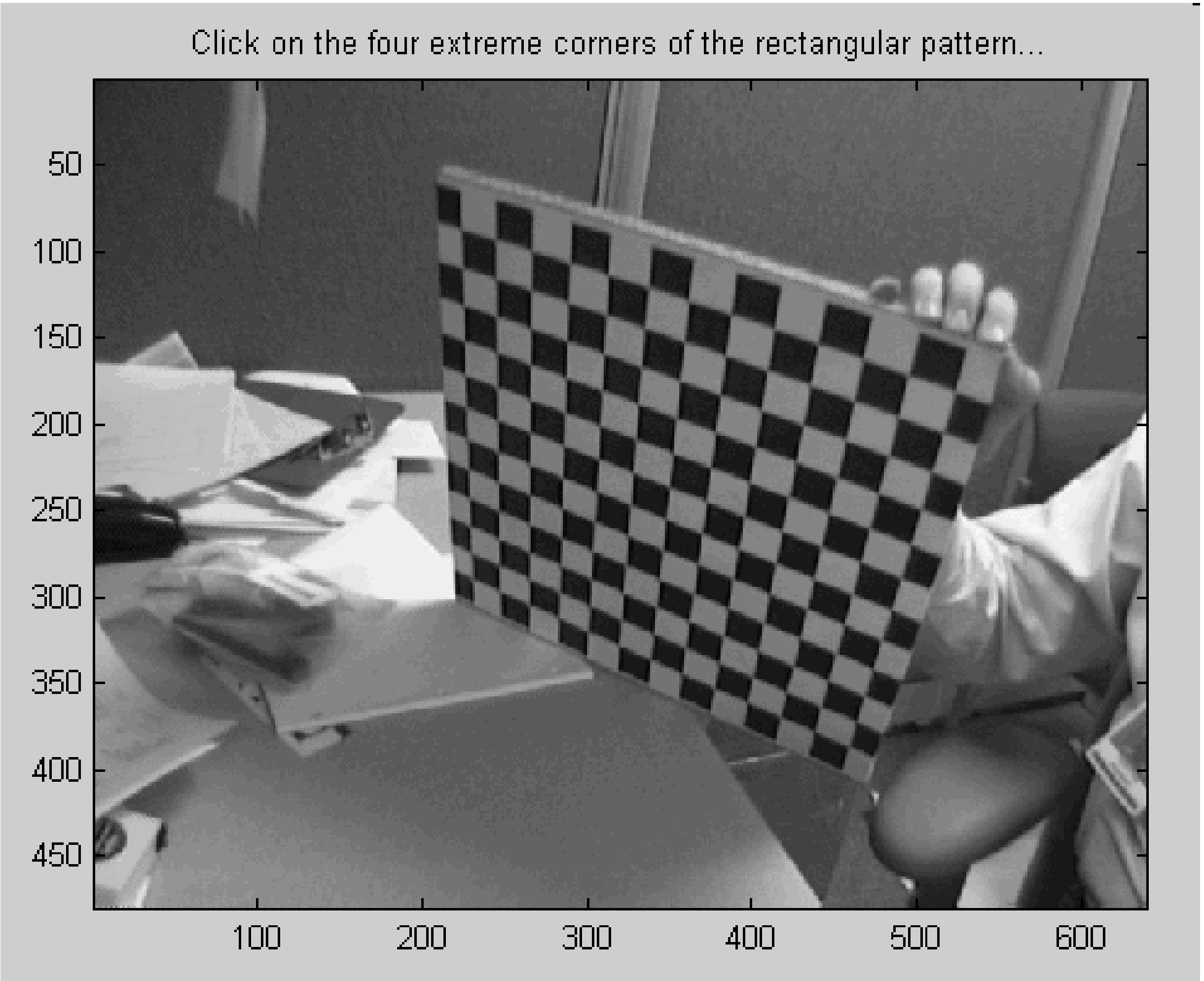
- zero-skew, square pixel
- $u_o$, $v_o$ = known center of the image
- no distortion

→ Just estimate f and R, T

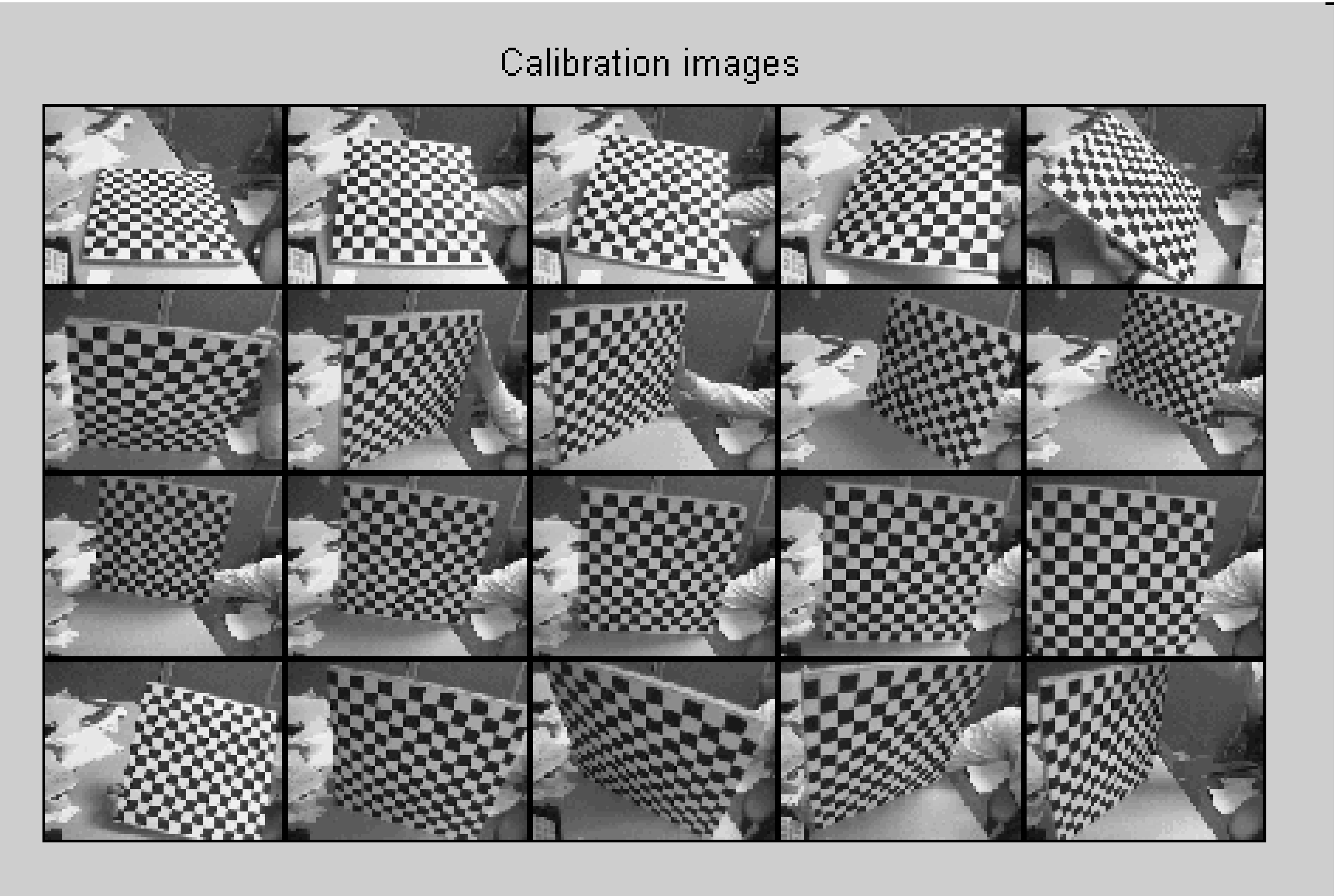
# Calibration Procedure

*Camera Calibration Toolbox for Matlab*
*J. Bouguet* – [1998-2000]

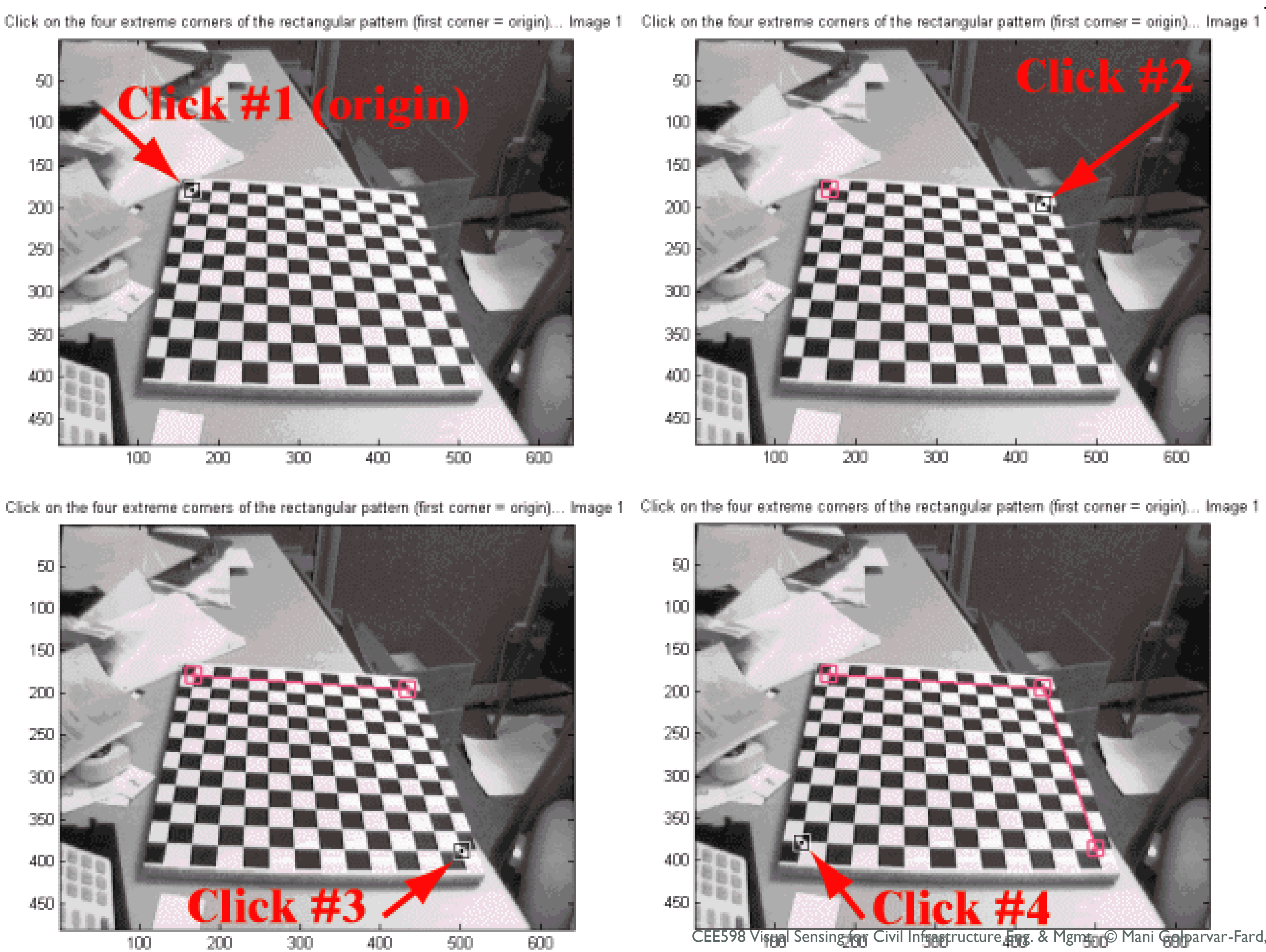
**http://www.vision.caltech.edu/bouguetj/calib_doc/index.html#examples**



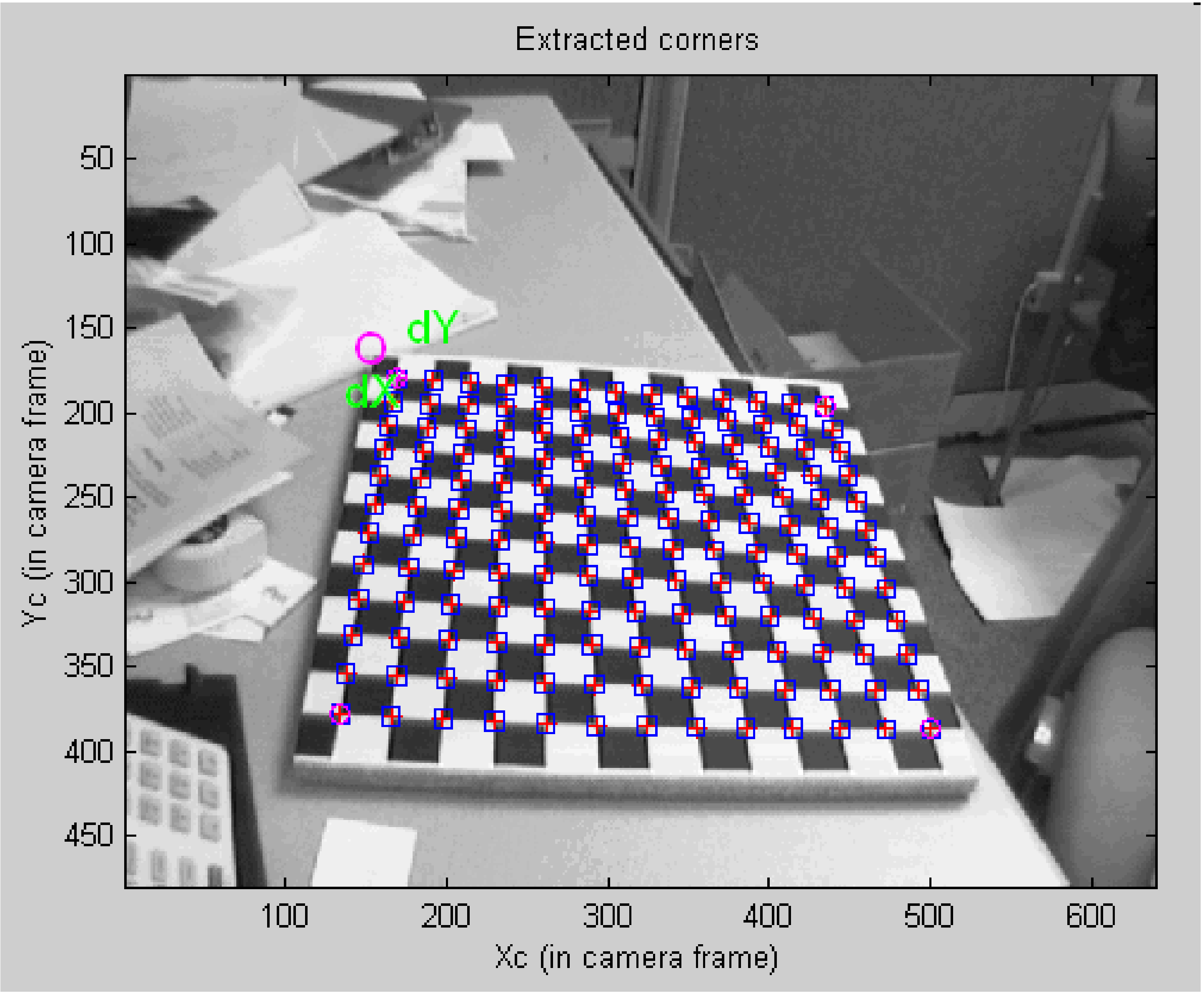
# Calibration Procedure



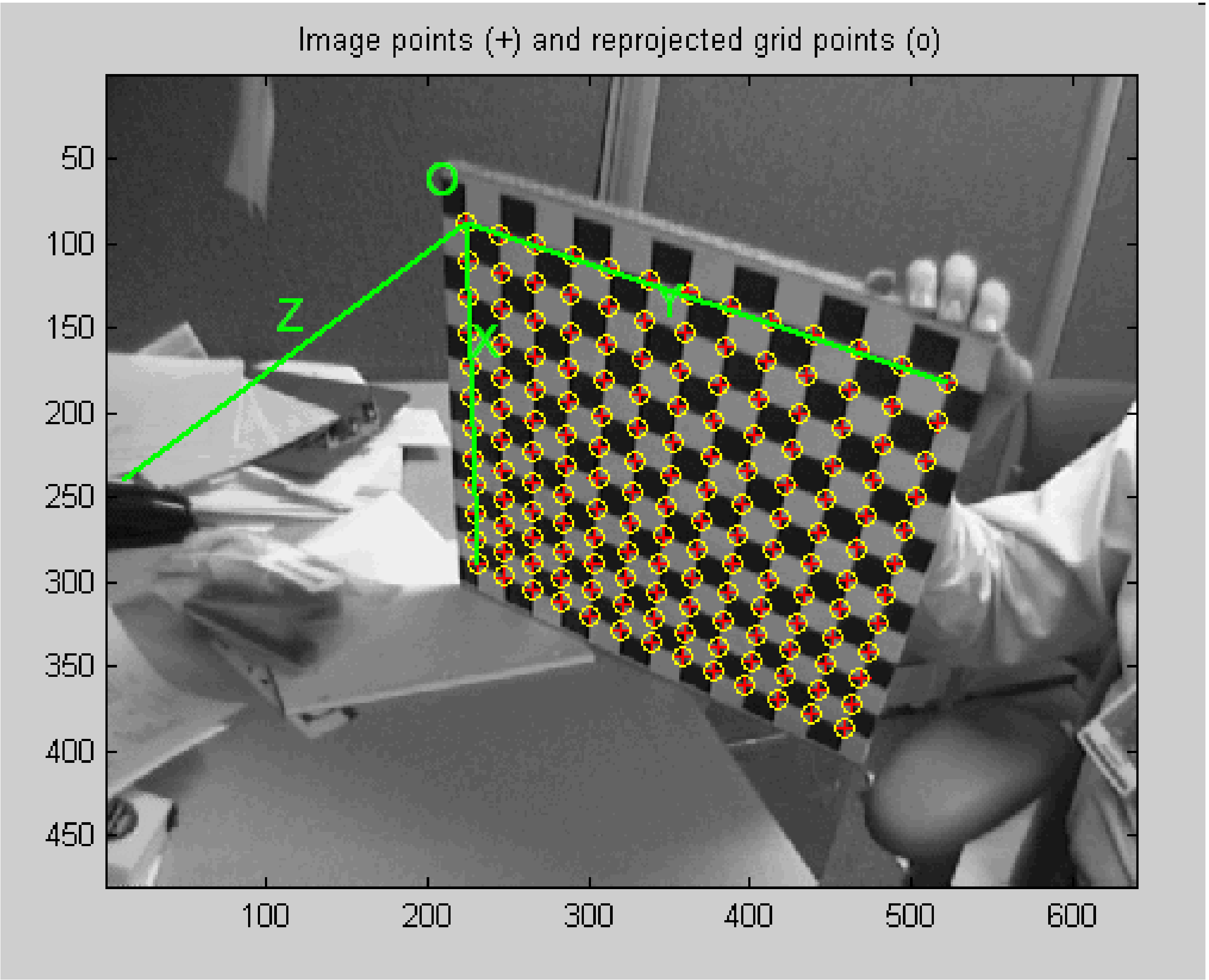
# Calibration Procedure



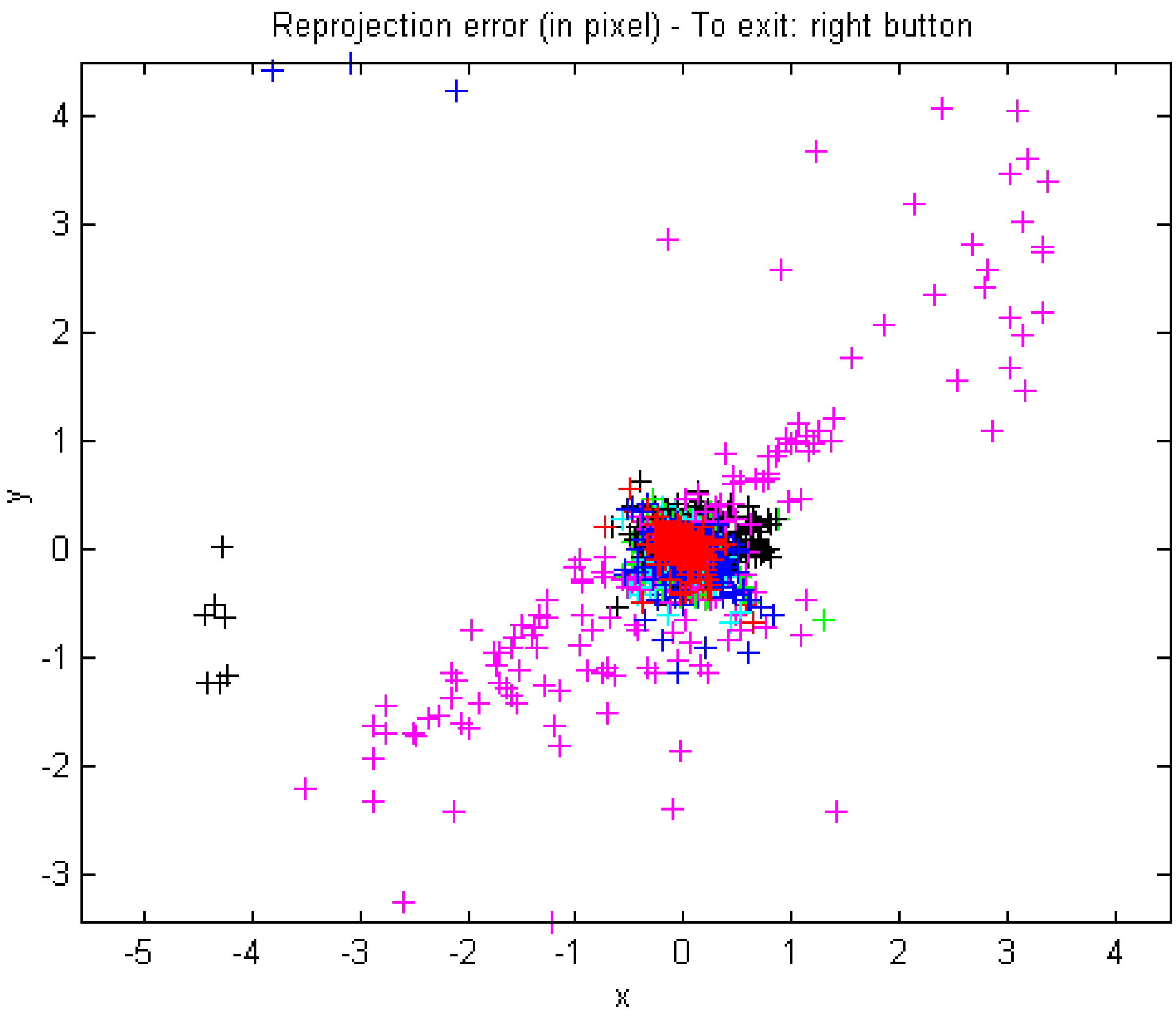
# Calibration Procedure



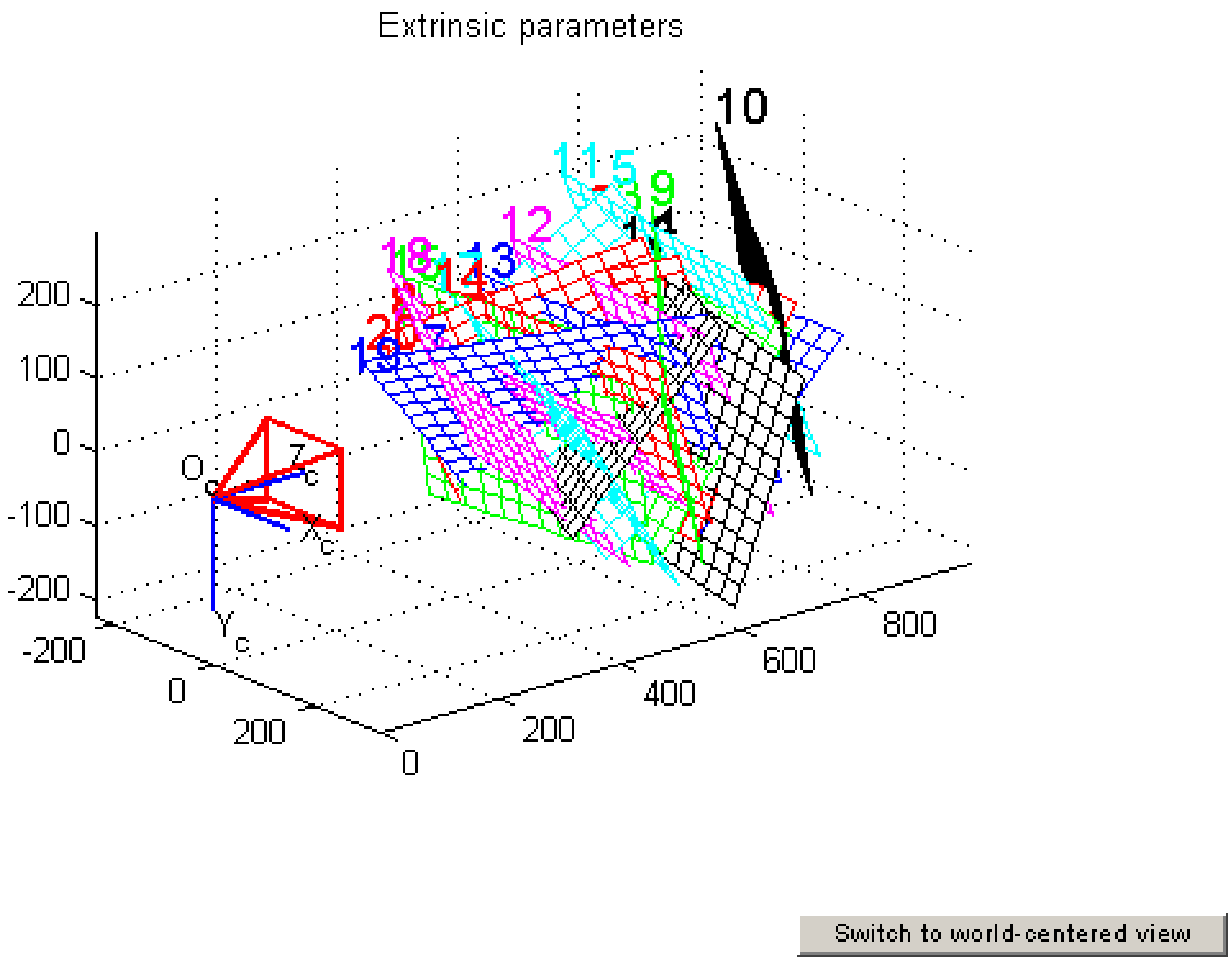
# Calibration Procedure



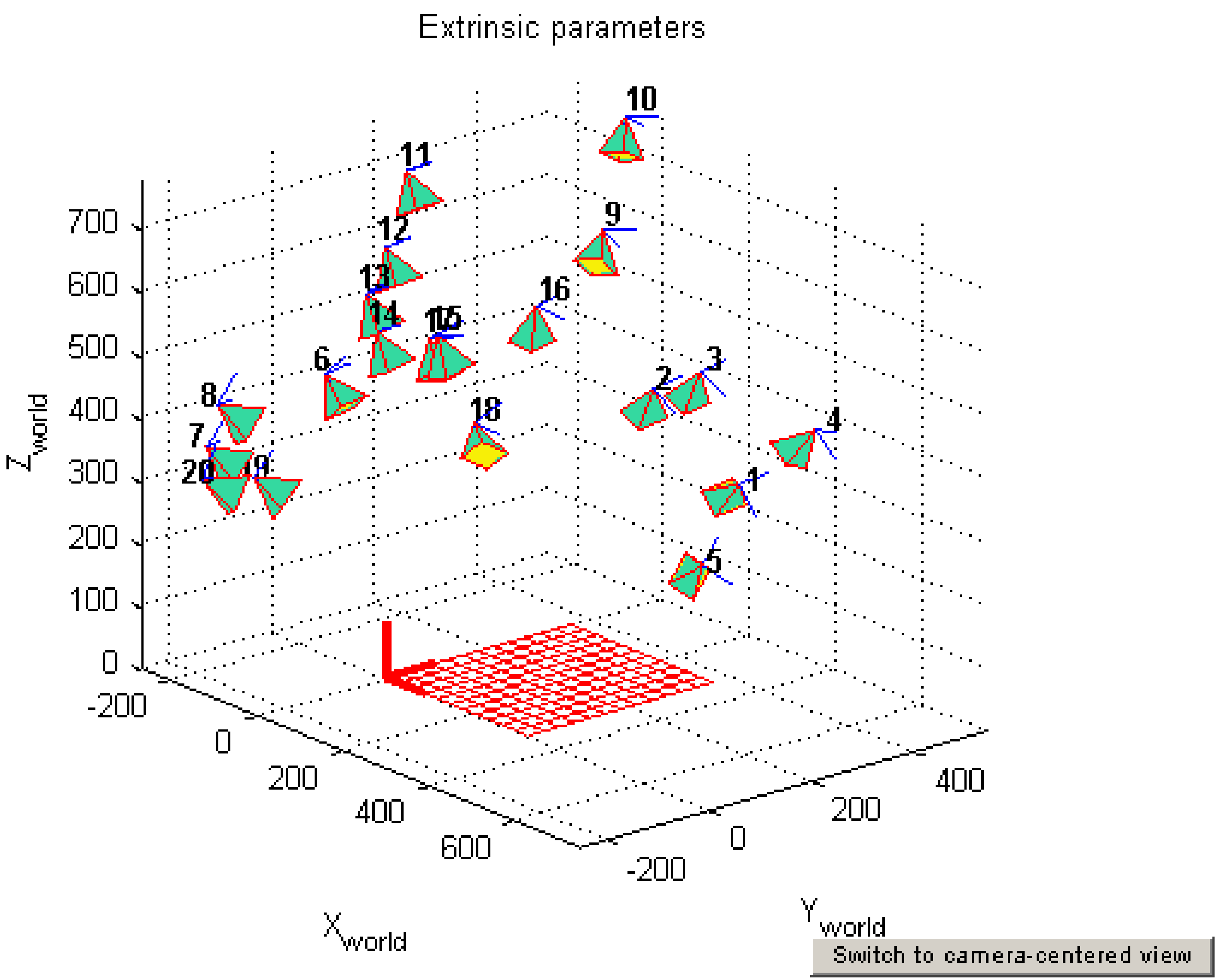
# Calibration Procedure



# Calibration Procedure



# Calibration Procedure



# Next lecture

- Single view reconstruction

- Assignment 1 will be online Next Tuesday